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Monopoles, noncommutative gauge theories in the BPS limit and some simple gauge groups

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Abstract: For three conspicuous gauge groups, namely, SU(2), SU(3) and SO(5), and at first order in the noncommutative parameter matrix $h\theta^{\mu\nu}$, we construct smooth monopole — and, some two-monopole — fields that solve the noncommutative Yang-Mills-Higgs equations in the BPS limit and that are formal power series in $h\theta^{\mu\nu}$. We show that there exist noncommutative BPS (multi-)monopole field configurations that are formal power series in $h\theta^{\mu\nu}$ if, and only if, two a priori free parameters of the Seiberg-Witten map take very specific values. These parameters, that are not associated to field redefinitions nor to gauge transformations, have thus values that give rise to sharp physical effects.

Keywords: Solitons Monopoles and Instantons, Nonperturbative Effects.

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1. Introduction

Although they have not been detected at the laboratory yet, monopoles play a key role in the understanding of some properties of non-abelian gauge theories. In QCD, where monopole degrees of freedom are uncovered by means of the Abelian projection, the confinement of colour can be explained as the effect of monopole condensation in the vacuum [1]. Monopoles, namely, BPS monopoles, occur as single-particle states in quantum non-abelian gauge theories with extended supersymmetry (see ref. [2] and references therein). S-duality — the generalization of the Montonen-Olive electric-magnetic duality conjecture — seems to be realized in $N = 4$ super-Yang-Mills theory and some $N = 2$ supersymmetric gauge theories with vanishing β-function (for further information the reader is referred to refs. [3, 4]).

BPS monopoles have been constructed and studied for some noncommutative $\mathrm{U}(N)$ gauge theories [5 − 11]. In particular, in refs. [6] and [8], a noncommutative $\mathrm{U}(2)$ BPS monopole was explicitly constructed up to second order in the noncommutative parameters $\theta^{\mu\nu}$ by expanding the BPS equations in powers of these parameters. The monopole so obtained is smooth and goes to the ordinary SU(2) BPS monopole as $\theta^{\mu\nu} \to 0$. And yet, up to the best of our knowledge, no results concerning the existence and no explicit construction of monopoles are available so far for noncommutative gauge theories with simple gauge groups such as SU(N) or SO(N). It is the main purpose of this paper to look for and give explicit monopole — and some two-monopole — solutions to the noncommutative equations of motion for noncommutative Yang-Mills-Higgs theories in the BPS limit when the gauge groups are SU(2), SU(3) and SO(5). Let us next argue why we have chosen SU(2), SU(3) and SO(5) as gauge groups.

It has long been known [12] that in ordinary Yang-Mills-Higgs theories with simple gauge groups and when there is maximal symmetry breaking, all magnetically charged BPS solutions may be regarded as multi-monopole configurations containing suitable numbers of different types of the so-called fundamental monopoles. The fundamental monopoles of the theory are obtained by embedding the SU(2) BPS monopole in the SU(2) subgroups of the gauge group of the theory furnished by its simple roots. Hence it seems natural to start out by constructing monopole solutions for noncommutative gauge theories with gauge group SU(2). Once this is done we would like to see how things work for larger simple gauge groups. The simplest choice seems to be SU(3). Next, when the gauge symmetry is not broken to the maximal torus of the gauge group, but the unbroken gauge group has a non-Abelian component, there exist degrees of freedom that show the presence of massless monopoles [13]. These massless monopoles do not occur classically as isolated solutions to the BPS equations and must be studied as part of multi-monopole configurations. The simplest instance of a theory where the existence of these massless monopoles can be analysed was furnished in ref. [13]: it is a theory with gauge group SO(5) broken down to SU(2) $\times$ U(1).

To formulate a noncommutative field theory whose gauge group is SU(N), there is only one available framework. This is the formalism put forward in refs. [14, 15] that led to the formulation of the noncommutative standard model [16] and some Grand Unified theories [17]. The phenomenology [18 – 20] that these theories give rise to may be detected at the LHC.

In the formalism of refs. [14, 15] — that can be used for any representation of any gauge group — the noncommutative gauge fields are defined from the ordinary fields by means of the Seiberg-Witten map, this map being given by a formal power series in $\theta^{\mu\nu}$. The noncommutative gauge fields thus take values in the enveloping algebra of the Lie algebra of the gauge group. This is very much at variance with the standard formalism used in noncommutative gauge theory, which demands the gauge group to be U(N). Hence, unlike in the ordinary Minkowski space-time case, the noncommutative Yang-Mills-Higgs theories to be considered in this paper are not theories that are part of the U(N) theories analysed in refs. [5 – 10]

The layout of this paper is as follows. In section 2 we define our noncommutative Yang-Mills-Higgs theories and the asymptotic behaviour of the fields. We also discuss the Bogomol'nyi bound and deduce the noncommutative BPS equations. The computation of the most general monopole solution — when it exists — to the noncommutative SU(2) BPS equations at first order in $\theta^{\mu\nu}$ is carried out in section 3. In this section, we also discuss the existence of noncommutative fundamental BPS monopoles and some two-monopoles for SU(3) and, finally, the existence of solutions to the noncommutative BPS equations that correspond to the family of solutions with a massless monopole reported in ref. [13] for SO(5). Since, in general, the noncommutative BPS equations studied in section 3 have no solutions that are formal power series in $\theta^{\mu\nu}$, we compute in section 4 the static solutions to the noncommutative Yang-Mills-Higgs equations with vanishing Higgs potential which go to the ordinary BPS monopole solutions for SU(2) and to the fundamental and two-monopoles considered previously for SU(3). The computations are carried out in the gauge $a_0 = 0$. How the noncommutative character of space-time affects at first order in $\theta^{\mu\nu}$ the SO(5) family of solutions with massless monopoles displayed in ref. [13] is also studied here. In the appendix, we discuss whether or not Derrick's theorem implies — as does in the instanton case, see ref. [21] — that there are no solutions at second order in $\theta^{\mu\nu}$ to the noncommutative Yang-Mills-Higgs equations solved in section 4.

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2. The noncommutative Hamiltonian, Bogomol'nyi bounds and the noncommutative BPS equations

Our noncommutative gauge theories will have the following action

$$S = \int d^4x \; -\frac{1}{2}\mathrm{Tr}F_{\mu\nu}\star F^{\mu\nu} + \mathrm{Tr}(D_\mu\Phi)^\dagger \star D^\mu\Phi. \tag{2.1}$$

The symbol $\star$ will stand for the Moyal product: $(f\star g)(x) = f(x)e^{\frac{i}{2}\,h\,\theta^{\mu\nu}\overleftarrow{\partial_\mu}\overrightarrow{\partial_\nu}}g(x)$. The noncommutative field strength $F_{\mu\nu}$ and the covariant derivative D_μ are given by $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$, $D_\mu = \partial_\mu - i[A_\mu, \quad]_\star$, respectively. A_μ and Φ denote the noncommutative gauge field and the Higgs field, respectively. They are defined in terms of the ordinary gauge field, a_μ, and the ordinary Higgs field, ϕ, by means of the Seiberg-Witten map, which we shall take to be a formal power series in $h\theta^{\mu\nu}$. The ordinary fields a_μ and ϕ take values in the Lie algebra of the gauge group — in our case, SU(2), SU(3) and SO(5) — in the fundamental representation. We shall normalize the generators of the gauge group, the hermitian matrices T^a, as follows $\mathrm{Tr}\,T^aT^b = \frac{1}{2}\,\delta^{ab}$, and assume that there is a dimensionful parameter v in the theory defined by $v = 2\mathrm{Tr}\,\phi^2(t, |\vec{x}| \to \infty)$.

The Seiberg-Witten map is not unique — a fact very much welcomed when proving renormalizability of some models [22, 23]. At first order in $h\theta^{\mu\nu}$ the most general expression for it that yields hermitian noncommutative fields and is a polynomial in the fields, their derivatives and v — we want the map to be well-defined when v vanishes — reads

$$\begin{aligned}
A_\mu &= a_\mu - \frac{h}{4}\theta^{\alpha\beta}\{a_\alpha, \partial_\beta a_\mu + f_{\beta\mu}\} + hD_\mu H + hS_\mu + O(h^2),\\
\Phi &= \phi - \frac{h}{4}\theta^{\alpha\beta}\{a_\alpha, 2D_\beta\phi + i[a_\beta, \phi]\} + ih[H, \phi] + hF + O(h^2),\\
H &= \mu_1\,\theta^{\alpha\beta}f_{\alpha\beta} + \mu_2\,\theta^{\alpha\beta}[a_\alpha, a_\beta],\\
S_\mu &= \kappa_1\,\theta^{\alpha\beta}D_\mu f_{\alpha\beta} + \kappa_2\,\theta_\mu{}^\beta\{D_\beta\phi, \phi\} + i\kappa_3\theta_\mu{}^\beta\,[D_\beta\phi, \phi] + k_4\,v\,\theta_\mu{}^\beta D_\beta\phi + w\,\theta_\mu{}^\rho D^\nu f_{\nu\rho},\\
F &= \lambda_1\theta^{\alpha\beta}\,\{f_{\alpha\beta}, \phi\} + i\lambda_2\,\theta^{\alpha\beta}[f_{\alpha\beta}, \phi] + \lambda_3\,v\,\theta^{\alpha\beta}f_{\alpha\beta}.
\end{aligned} \tag{2.2}$$

The symbols μ_i, κ_i, λ_i and w denote dimensionless real constants, v is the parameter with mass dimension defined above, $f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu - i[a_\mu, a_\nu]$ and $D_\mu = \partial_\mu - i[a_\mu, \quad]$. When all the constants μ_i, κ_i, λ_i and w are set to zero, one gets the standard Seiberg-Witten map, i.e., the straightforward generalization to our case of the map originally introduced by Seiberg and Witten for $\mathrm{U}(1)$ noncommutative gauge theories. Notice that the monomials $\kappa_1\,\theta^{\alpha\beta}D_\mu f_{\alpha\beta}$, $\kappa_3\theta_\mu{}^\beta\,[D_\beta\phi,\phi]$, $w\,\theta_\mu{}^\rho D^\nu f_{\nu\rho}$, $i\lambda_2\,\theta^{\alpha\beta}[f_{\alpha\beta},\phi]$, $\kappa_4\,v\,\theta_\mu{}^\beta D_\beta\phi$ and $\lambda_3\,v\,\theta^{\alpha\beta}f_{\alpha\beta}$ always belong to the Lie algebra of the simple gauge group and thus can be set to zero by redefining the field a_μ. However, the terms $\kappa_2\,\theta_\mu{}^\beta\{D_\beta\phi,\phi\}$ and $\lambda_1\theta^{\alpha\beta}\,\{f_{\alpha\beta},\phi\}$ do not belong to the Lie algebra of the simple gauge group and hence they do not correspond to field redefinitions of a_μ. The terms in eq. (2.3) that go with H are gauge transformations. Notice that in the noncommutative $\mathrm{U}(N)$ case of refs. [5 − 11] the terms $\kappa_2\,\theta_\mu{}^\beta\{D_\beta\phi,\phi\}$ and $\lambda_1\theta^{\alpha\beta}\,\{f_{\alpha\beta},\phi\}$ also correspond to field redefinitions of a_μ. We shall see in the next section that at least for SU(2), SU(3) and SO(5), and at odds with the $\mathrm{U}(N)$ case, the value of the real constants κ_2 and λ_1 is physically relevant.

In this paper we will not be interested in the most general Seiberg-Witten map. Indeed, in keeping with the situation for the noncommutative $\mathrm{U}(N)$ theories of refs. [5 − 10], we shall restrict ourselves to theories whose action in the temporal gauge — here, $a_0 = 0$ — depends on the generalized coordinates — $a_i(t,\vec{x})$ and $\phi(t,\vec{x})$, in our case —, the generalized velocities — $\partial_0 a_i(t,\vec{x})$ and $\partial_0\phi(t,\vec{x})$, for our theories — and the spatial derivatives of them, but not on generalized accelerations nor on any other higher time derivatives. Thus, the noncommutative matrix parameter $\theta^{\mu\nu}$ will be taken to be of magnetic type — i.e., $\theta^{0i} = 0$ — and $\Phi[\phi, a_\mu]$ and $A_i[a_\mu,\phi]$ must not involve time derivatives — otherwise $D_0\Phi$ or F_{0i} in eq. (2.1) would give rise, at least, to second order time derivatives. $a_0 = 0$ does not imply $A_0 = 0$, but restricts the form of A_0 to linear combinations of terms linear in $(\partial_0 a_i, \partial_0\phi)$, the coefficients of these combinations being functions of the ordinary fields and/or their spatial derivatives, but having no time derivatives of the former. For this Seiberg-Witten map, F_{ij} and $D_i\Phi$ do not involve time derivatives of ordinary fields, and F_{0i} and $D_0\Phi$ are linear combinations of terms proportional to $\partial_0 a_i, \partial_0\phi$, with coefficients free of time derivatives. That a Seiberg-Witten map — in fact, infinitely many — satisfying the previous requirements exists at any order in $h\theta^{\mu\nu}$ can be readily shown by using the Seiberg-Witten map defined by the following equations:

$$\begin{aligned}
\frac{dA_\mu}{dh} &= -\frac{1}{4}\theta^{ij}\{A_i, \partial_j A_\mu + F_{j\mu}\}_\star + D_\mu\hat{H} + \hat{S}_\mu \\
\frac{d\Phi}{dh} &= -\frac{1}{4}\theta^{ij}\{A_i, 2D_j\Phi + i[A_j,\Phi]_\star\}_\star + i[\hat{H},\Phi]_\star + \hat{F} \\
\hat{H} &= \mu_1\,\theta^{ij}F_{ij} + \mu_2\,\theta^{ij}[A_i,A_j]_\star, \\
\hat{S}_\mu &= \kappa_1\,\theta^{ij}D_\mu F_{ij} + \kappa_2\,\theta_\mu{}^j\{D_j\Phi,\Phi\}_\star + i\kappa_3\theta_\mu{}^j\,[D_j\Phi,\Phi]_\star + \kappa_4\,v\,\theta_\mu{}^j D_j\Phi, \\
\hat{F} &= \lambda_1\theta^{ij}\,\{F_{ij},\Phi\} + i\lambda_2\,\theta^{ij}[F_{ij},\Phi]_\star + \lambda_3\,v\,\theta^{ij}F_{ij},
\end{aligned} \tag{2.3}$$

where μ_i, κ_i and λ_i are dimensionless real constants, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]_\star$ and $D_\mu = \partial_\mu - i[A_\mu, \quad]_\star$.

The restrictions imposed on the Seiberg-Witten map in the previous paragraph do not give a unique Seiberg-Witten map, though. At first order in $h\theta^{\mu\nu}$, they merely set $w = 0$.

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However, this yields an action that is quadratic in the generalized velocities so that the Hamiltonian can be derived from it by using the standard textbook formalism. Indeed, there is a single generalized momenta, $p_i = \frac{\partial\mathcal{L}}{\partial\dot{q}_i}$, per generalized coordinate, q_i, and the Hamiltonian, $\mathcal{H}$, can be obtained from the Lagrangian, $\mathcal{L}$, by employing the elementary expression $\mathcal{H} = \sum_i p_i\dot{q}_i - \mathcal{L}$. In our case the Hamiltonian reads

$$\mathcal{H} = \int d^3\vec{x}\ \mathrm{Tr}\left(E_iE_i + B_iB_i + D_0\Phi D_0\Phi + D_i\Phi D_i\Phi\right), \tag{2.4}$$

where $E_i = F_{i0}$, $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$. The Hamiltonian has been computed in the gauge $a_0 = 0$; the Gauss-law constraint takes here the form

$$\mathrm{Tr}\frac{\delta A_0}{\delta a_0^a}\left(D_jE_j + i[D_0\Phi, \Phi]_\star\right) = 0. \tag{2.5}$$

Let us note that although the Hamiltonian in eq. (2.4) is defined by the same formal expression as in the $\mathrm{U}(N)$ case of refs. [5 − 10], the Gauss-Law constraint signals a difference with the $\mathrm{U}(N)$ case, where it reads $D_jE_j + i[D_0\Phi, \Phi]_\star = 0$. This difference stems from the fact that for simple gauge groups, unlike for $\mathrm{U}(N)$ gauge groups, noncommutative fields do not take values in the Lie algebra of the gauge group.

We shall introduce next the asymptotic boundary conditions for the noncommutative fields $\Phi(t,\vec{x})$ and $A_\mu(t,\vec{x})$. These conditions read

$$\begin{aligned}
\Phi(t,\vec{x}) &\sim \phi(t,\vec{x}) + O\left(\tfrac{1}{|\vec{x}|^2}\right) \qquad &&\text{as} \qquad |\vec{x}| \to \infty,\\
A_\mu(t,\vec{x}) &\sim a_\mu(t,\vec{x}) + O\left(\tfrac{1}{|\vec{x}|^2}\right) \qquad &&\text{as} \qquad |\vec{x}| \to \infty.
\end{aligned} \tag{2.6}$$

A simple dimensional analysis shows that the asymptotic boundary conditions above follow from the Seiberg-Witten map defined at first order by eq. (2.3) — and at higher-order by eq. (2.4) — and the asymptotic boundary conditions for the ordinary fields that we set next. For the ordinary fields $\phi(t,\vec{x})$ and $a_i(t,\vec{x})$, we shall take the boundary conditions in the gauge $a_0 = 0$ that are customary in monopole physics [24]:

$$\begin{aligned}
\phi(t,\vec{x}) &= g(t,\hat{x})\phi_0 g(t,\hat{x})^\dagger + O\left(\tfrac{1}{|\vec{x}|}\right) \qquad &&\text{as} \qquad |\vec{x}| \to \infty,\\
a_i(t,\vec{x}) &\sim \tfrac{1}{|\vec{x}|} &&\text{as} \qquad |\vec{x}| \to \infty,\\
D_i\phi &\sim \tfrac{1}{|\vec{x}|^2} &&\text{as} \qquad |\vec{x}| \to \infty,
\end{aligned} \tag{2.7}$$

where $\hat{x} = \vec{x}/|\vec{x}|$ and ϕ_0 is the value of the Higgs field along a given fixed direction in space. $g(t,\hat{x})$ defines a smooth map from the two-sphere at spatial infinity into the coset G/H, G and H being respectively the broken and unbroken gauge groups.

Let us introduce now the magnetic charge, Q_M, of the noncommutative fields:

$$Q_M = \frac{1}{2\pi v}\mathrm{Tr}\int dS_i\ B_i\Phi = \frac{1}{2\pi v}\mathrm{Tr}\int dS_i\ b_i\phi. \tag{2.8}$$

b_i and ϕ are the ordinary field configurations that yield B_i and Φ upon acting with the Seiberg-Witten map. The integrals are carried out over a two-sphere at spatial infinity and $v = (2\,\mathrm{Tr}\,\phi^2(|\vec{x}| \to \infty))^{\frac{1}{2}} = (2\,\mathrm{Tr}\,\phi_0^2)^{\frac{1}{2}}$. Q_M depends only on the boundary conditions for the fields. The equality between the two surface integrals above follows from the asymptotic boundary conditions in eq. (2.6) and in turn implies that the noncommutative fields carry the same magnetic charges as the BPS (multi-)monopoles of the corresponding ordinary theory. Indeed, both the boundary conditions in eq. (2.7) and the form of the Seiberg-Witten map in eqs. (2.3) and (2.4) lead to the conclusion that at very large distances the chief contributions to the equations of motion of our noncommutative theory are given by the corresponding ordinary Yang-Mills-Higgs equations. Of course, Q_M above is constrained by the quantization condition of ref. [25].

Let us apply now the Bogomol'nyi trick to the r.h.s. of eq. (2.4):

$$\mathcal{H} = \int d^4x\,\mathrm{Tr}\left(D_0\Phi D_0\Phi + E_iE_i + (B_i \mp D_i\Phi)^2 \pm 4\pi\, v\, Q_M\right) \geq 4\pi\, v\,|Q_M|. \tag{2.9}$$

Hence, for each value of Q_M — as in the ordinary case — , the absolute minima of the energy are given by the solutions to the equations

$$B_i = \pm D_i\Phi, \quad D_0\Phi = 0, \quad E_i = 0. \tag{2.10}$$

These equations are the noncommutative BPS equations. Notice that they are the straightforward generalization to noncommutative space-time of the ordinary BPS equations. Also notice that the noncommutative BPS equations above imply the Gauss-law constraint in eq. (2.5).

That the meaning and form of the noncommutative BPS equations is analogous to those of the ordinary BPS equations and that the magnetic charge of the noncommutative field configurations is the same as that of their ordinary counterparts are facts that our theories share in common with the $\mathrm{U}(N)$ noncommutative theories studied in ref. [5, 6, 8]. However, we shall see in the next section that the BPS moduli spaces of our theories are quite different from the corresponding spaces of the $\mathrm{U}(N)$ case.

To close this section let us point out that the solutions to the noncommutative BPS equations in eq. (2.10) are also solutions to the Yang-Mills-Higgs equations derived from the action in eq. (2.1). The latter equations read

$$\begin{aligned}
&\int d^4x \left\{ \mathrm{Tr}\left[\frac{\delta A_\nu(x)}{\delta a^a_\mu(y)}\{D_\rho F^{\rho\nu}(x) - i[\Phi, D^\nu\Phi]_\star(x)\}\right] - \mathrm{Tr}\left[\frac{\delta\Phi(x)}{\delta a^a_\mu(y)}\{D_\rho D^\rho\Phi(x)\}\right]\right\} = 0\\
&\int d^4x \left\{ \mathrm{Tr}\left[\frac{\delta A_\nu(x)}{\delta\phi^a(y)}\{D_\rho F^{\rho\nu}(x) - i[\Phi, D^\nu\Phi]_\star(x)\}\right] - \mathrm{Tr}\left[\frac{\delta\Phi(x)}{\delta\phi^a(y)}\{D_\rho D^\rho\Phi(x)\}\right]\right\} = 0.
\end{aligned} \tag{2.11}$$

3. Solutions to the noncommutative BPS equations

In this section we shall look for solutions to the BPS equations given in eqs. (2.10) that are formal power series in $h\theta^{\mu\nu}$. We shall work in the temporal gauge $a_0 = 0$ and consider the following (broken) gauge groups: SU(2), SU(3) and SO(5). These groups will be

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broken down to U(1), U(1) × U(1) and SU(2) × U(1), respectively, by choosing appropriate asymptotic boundary conditions for the Higgs field.

Let us recall — see previous section— that our Seiberg-Witten map — for $a_0 = 0$ — is such that A_0 is linear in ($\dot{a}^a_i = \partial_0 a^a_i, \dot{\phi}^a = \partial_0 \phi^a$) with coefficients that are constructed only with a_i, ϕ and ∂_k, and that A_i and Φ only depend on a_i, ϕ and their spatial partial derivatives. Then,

$$\begin{aligned} A_0 &= \sum_{l>0} h^l L_0^{(l)ia}[\theta^{\mu\nu}, a_k, \phi, \partial_k]\dot{a}^a_i + \sum_{l>0} h^l M_0^{(l)a}[\theta^{\mu\nu}, a_k, \phi, \partial_k]\dot{\phi}^a \\ F_{0i} &= \dot{a}_i + \sum_{l>0} h^l P_{0i}^{(l)ja}[\theta^{\mu\nu}, a_k, \phi, \partial_k]\dot{a}^a_j + \sum_{l>0} h^l Q_{0i}^{(l)a}[\theta^{\mu\nu}, a_k, \phi, \partial_k]\dot{\phi}^a \\ D_0\Phi &= \dot{\phi} + \sum_{l>0} h^l S_0^{(l)ja}[\theta^{\mu\nu}, a_k, \phi, \partial_k]\dot{a}^a_j + \sum_{l>0} h^l T_0^{(l)a}[\theta^{\mu\nu}, a_k, \phi, \partial_k]\dot{\phi}^a, \end{aligned}$$

where $L_0^{(l)ia}$, $M_0^{(l)a}$, $P_{0i}^{(l)ja}$, $Q_{0i}^{(l)a}$, $S_0^{(l)ja}$ and $T_0^{(l)a}$ are homogeneous polynomials in $\theta^{\mu\nu}$ of degree l. The previous expressions lead to the conclusion that if a_i and ϕ are formal power series in $h\theta^{\mu\nu}$, the following result holds

$$E_i = 0 \quad \text{and} \quad D_0\Phi = 0 \quad \Longleftrightarrow \quad \dot{a}_i = 0 \quad \text{and} \quad \dot{\phi} = 0.$$

Hence, in the remainder of this section, we shall look for solutions to $B_i = \pm D_i\Phi$ that are time independent and are given by the following formal expansions in powers of $h\theta^{\mu\nu}$:

$$a_i = a_i^{(0)} + \sum_{l>0} h^l a_i^{(l)}, \quad \phi = \phi^0 + \sum_{l>0} h^l \phi^{(l)}. \tag{3.1}$$

$a_i^{(l)}$ and $\phi^{(l)}$ are homogeneous polynomials in $\theta^{\mu\nu}$ of degree l. We shall use besides the following power series in $h\theta^{\mu\nu}$:

$$f_{ij} = f_{ij}^{(0)} + \sum_{l>0} h^l f_{ij}^{(l)}, \quad D_k\phi = (D_k\phi)^0 + \sum_{l>0} h^l (D_k\phi)^{(l)}, \tag{3.2}$$

where $f_{ij}^{(0)} = \partial_i a_j^{(0)} - \partial_j a_i^{(0)} - i[a_i^{(0)}, a_j^{(0)}]$ and $(D_k\phi)^0 = \partial_k\phi^{(0)} - i[a_k^{(0)}, \phi^{(0)}]$, and $f_{ij}^{(l)}$ and $(D_k\phi)^{(l)}$ are also homogeneous polynomials in $\theta^{\mu\nu}$ of degree l.

3.1 SU(2) noncommutative BPS magnetic (anti-)monopoles

Let us seek for time-independent a_i and ϕ that belong to the Lie algebra of SU(2) in its fundamental representation and that solve $B_i = \pm D_i\Phi$ at first order in $h\theta^{\mu\nu}$. We shall further assume that the asymptotic boundary conditions are such that $Q_M = \pm 1$ — see eq. (2.8), i.e., we shall look for noncommutative BPS monopoles and anti-monopoles.

We shall begin our analysis by assuming that the noncommutative fields are defined by the standard form of the Seiberg-Witten map. This form is obtained by setting $\hat{H}, \hat{S}_\mu$ and $\hat{F}$ in eq. (2.4) to zero. For the standard form of the Seiberg-Witten map in the gauge $a_0 = 0$ and for time-independent field configurations, it is easy to see that the field Φ is defined by the standard form of the Seiberg-Witten map that corresponds to the A_4

component of the gauge field in a noncommutative space-time with Euclidean signature. Hence, we can combine a_i and ϕ into an Euclidean ordinary gauge field $\overline{a}_\mu = (a_i, a_4 = \phi)$ and A_i and Φ into a noncommutative gauge field $\overline{A}_\mu = (A_i, A_4 = \Phi)$, so that, again, the standard form of the Seiberg-Witten map defines $\overline{A}_\mu$ in terms of $\overline{a}_\mu$. Now, with the definition $\overline{F}_{\mu\nu} = \partial_\mu \overline{A}_\nu - \partial_\nu \overline{A}_\mu - i[\overline{A}_\mu, \overline{A}_\nu]_\star$ and recalling that neither a_μ, nor A_μ, depend on x^4, one concludes that the BPS equations in eq. (2.10) can be turned into the following (anti-)self-duality equations:

$$\overline{F}_{\mu\nu} = \pm \tilde{\overline{F}}_{\mu\nu}, \quad \tilde{\overline{F}}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\overline{F}_{\rho\sigma}.$$

Unfortunately, it has been shown in ref. [21] that even at first order in $h\theta^{\mu\nu}$ there are no solutions to the previous equation. There are thus no noncommutative (anti-)monopoles arising from the noncommutative SU(2) BPS equations for the standard form of the Seiberg-Witten map. Hence, all that remains for us to do is to see whether or not this negative result can be turned into a positive one by taking advantage of the ambiguities in the form of the Seiberg-Witten map that do not correspond neither to field redefinitions nor to gauge transformations.

For the general form — with $w = 0$ — of the Seiberg-Witten map given in eq. (2.3), the previous construction, that turns the BPS equations into the (anti-)self-duality equations above, cannot be carried out. Hence, we have to deal with the equation $B_i = \pm D_i\Phi$ directly. At zero order in $h\theta^{\mu\nu}$, the previous equation is the ordinary equation:

$$b_i^{(0)} = \pm (D_i\phi)^{(0)}, \tag{3.3}$$

where $b_i^{(0)} = \frac{1}{2}\epsilon_{ijk} f_{jk}^{(0)}$ and $(D_i\phi)^{(0)} = \partial_i\phi^{(0)} - i[a_i^{(0)}, \phi^{(0)}]$. $a_i^{(0)}$, $\phi^{(0)}$, $f_{ij}^{(0)}$ have been defined in eqs. (3.1) and (3.2).

The solutions to eq. (3.3) with magnetic charge ± 1 are the ordinary SU(2) (anti-)monopoles in the fundamental representation:

$$\begin{aligned} \phi^{(0)} &= \tfrac{x^a}{r} H(r)\, \tfrac{\sigma^a}{2}, & H(r) &= \pm\left(\tfrac{1}{r} - \lambda \coth \lambda r\right) \\ a_i^{(0)} &= [1 - K(r)]\, \epsilon_{ial}\, \tfrac{x^l}{r^2}\, \tfrac{\sigma^a}{2}, & K(r) &= 2 - \tfrac{\lambda r}{\sinh \lambda r}. \end{aligned} \tag{3.4}$$

where $\{\sigma^a\}_{\{a=1,2,3\}}$ stands for the Pauli matrices and $\lambda = v$ — later on we will consider SU(2) monopoles embedded in SU(3) and the value of λ will change.

The Seiberg-Witten map gives rise to the following expressions for the noncommutative objects F_{ij} and $D_k\Phi$ defined as power series in $h\theta^{\mu\nu}$:

$$F_{ij} = f_{ij} + \sum_{l>0} h^l F_{ij}^{(l)}[a_k, \phi], \quad D_k\Phi = D_k\phi + \sum_{l>0} h^l \mathcal{O}_k^{(l)}[\phi, a_i]. \tag{3.5}$$

Since a_i and ϕ are defined by the expansions in eq. (3.1), we end up with the following results

$$\begin{aligned} F_{ij}^{(l)}[a_k, \phi] &= \textstyle\sum_{m\geq 0} h^m F_{ij}^{(l,\,m)}, & F_{ij}^{(l,\,m)} &= \tfrac{1}{m!}\tfrac{d^m}{dh^m} F_{ij}^{(l)}[a_k, \phi]|_{h=0} \\ \mathcal{O}_k^{(l)}[\phi, a_k] &= \textstyle\sum_{m\geq 0} h^m \mathcal{O}_k^{(l,\,m)}, & \mathcal{O}_k^{(l,\,m)} &= \tfrac{1}{m!}\tfrac{d^m}{dh^m} \mathcal{O}_k^{(l)}[\phi, a_k]|_{h=0}. \end{aligned} \tag{3.6}$$

We are now ready to write down the contribution to $B_i = \pm D_i\Phi$ that is of order one in $h\theta^{\mu\nu}$:

$$(f_{ij}^{(1)} + F_{ij}^{(1,0)}) = \pm\epsilon_{ijk}[(D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}]. \tag{3.7}$$

The objects that occur in this equation have been defined in eqs. (3.1), (3.2), (3.5) and (3.6).

Both sides of eq. (3.7) take values in the universal enveloping algebra of SU(2) in the fundamental representation. Hence, both sides of eq. (3.7) can be expressed as a linear combination of the 2×2 identity matrix and the Pauli matrices. We thus conclude that eq. (3.7) is equivalent to the set of equations $a)$ and $b)$ that follow:

$$\begin{aligned} &a) \qquad \mathrm{Tr}\big[(f_{ij}^{(1)} + F_{ij}^{(1,0)})\big] = \pm\epsilon_{ijk}\mathrm{Tr}\big[(D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}\big],\\ &b) \qquad \mathrm{Tr}\big[\tfrac{\sigma^a}{2}(f_{ij}^{(1)} + F_{ij}^{(1,0)})\big] = \pm\epsilon_{ijk}\mathrm{Tr}\big[\tfrac{\sigma^a}{2}\big((D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}\big)\big]. \end{aligned} \tag{3.8}$$

Some little algebra leads to the result that $a)$ in eq. (3.8) is equivalent to

$$\begin{aligned} \sum_a \frac{1}{2}[(f_{12}^{(0),\,a})^2 + (f_{13}^{(0),\,a})^2 + (f_{23}^{(0),\,a})^2]\theta_{ij} &= \kappa_2\theta_{jk}\,\partial_i(\partial_k\phi^{(0),\,a}\phi^{(0),\,a}) - (i\leftrightarrow j)\\ &\quad \pm\lambda_1\epsilon_{ijk}\,\partial_k\,[\,\theta^{mn} f_{mn}^{(0),\,a}\phi^{(0),\,a}]. \end{aligned} \tag{3.9}$$

Since $a_i^{(0)} = a_i^{(0),\,a}\,\frac{\sigma^a}{2}$ and $\phi^{(0)} = \phi^{(0),\,a}\,\frac{\sigma^a}{2}$ are fixed by eq. (3.4), one concludes that eq. (3.9) — and hence $a)$ in eq. (3.8) — is more a no-go condition than an equation of motion. Indeed, it holds if, and only if, the parameters κ_2 and λ_1 of the Seiberg-Witten map in eq. (2.3) are tuned to the following values

$$\kappa_2 = -\frac{1}{2}, \quad \lambda_1 = \frac{1}{4}. \tag{3.10}$$

Next, taking into account the Seiberg-Witten map defined in eq. (2.3), one may show that the equality $b)$ in eq. (3.8) holds if, and only if,

$$D_i^{(0)}(a_j') - D_j^{(0)}(a_i') = \pm\epsilon_{ijk}\,\big(D_k^{(0)}\phi' - i[a_k', \phi^{(0)}]\big), \tag{3.11}$$

where

$$\begin{aligned} a_i' &= a_i^{(1)} + \kappa_1\,\theta^{kl}\,D_i^{(0)} f_{kl}^{(0)} + i\kappa_3\,\theta_i^{\;l}\,[(D_l\phi)^{(0)}, \phi^{(0)}] + \kappa_4\,v\theta_i^{\;j} D_j^{(0)}\phi^{(0)},\\ \phi' &= \phi^{(1)} + i\lambda_2\,\theta^{kl}\,[f_{kl}^{(0)}, \phi^{(0)}] + \lambda_3\,v\,\theta^{ij} f_{ij}^{(0)}. \end{aligned} \tag{3.12}$$

Now, eq. (3.11) is the equation of the zero modes associated to the ordinary SU(2) BPS (anti-)monopole. Hence, a_i', ϕ' in eq. (3.13) satisfy the zero mode equations in the background of the ordinary SU(2) BPS (anti-)monopole. Also notice that eq. (3.11) shows that the monomials $\kappa_2\,\theta_i^{\;j}\{D_j\phi, \phi\}$ and $\lambda_1\,\theta^{ij}\{f_{ij}, \phi\}$ do not contribute to $b)$ in eq. (3.8), so that the latter equations are not affected by the constraint in eq. (3.10). Let us stress that at first order in $h\theta^{\mu\nu}$ only for the choice of constants given in eq. (3.10) there exist BPS (anti-)monopole solutions to the noncommutative BPS equations defined with the help of the Seiberg-Witten map — with $w = 0$ — in eq. (2.3). These solutions are given

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by the ordinary (anti-)monopoles plus the field redefinitions that the terms of the Seiberg-Witten map which go with κ_1, κ_3 and λ_2 give rise to. From the previous statement one deduces that the terms $\kappa_2\,\theta_i^{\,j}\{D_j\phi,\phi\}$ and $\lambda_1\,\theta^{ij}\{f_{ij},\phi\}$ in eq. (2.3) that constitute part of the ambiguity in the Seiberg-Witten map — the other being field redefinitions and gauge transformations — are not physically irrelevant in the SU(2) case since the existence of a BPS moduli space with elements that are formal power series in $h\theta^{\mu\nu}$ depends drastically on the value of κ_2 and λ_1.

We shall close this subsection showing that the number of zero modes, or moduli, associated with the noncommutative BPS monopole found is four. Indeed, the noncommutative BPS equations are invariant under translations — three moduli-and the large gauge transformation $e^{i\chi\frac{\phi(\vec{x})}{v}}$, $0\leq\chi<2\pi$ — one moduli. One may rule out the possibility of the existence of further zero modes — that should vanish as $\theta^{\mu\nu}\rightarrow 0$ — as follows. Let $\delta z=(\delta a_i,\delta\phi)$ denote a zero mode that can be expressed as a power series in $h\theta^{\mu\nu}$: $\delta z=\sum_{l\geq 0}h^l\delta z^{(l)}$. Then the components of $\delta z^{(l)}$, which are homogeneous polynomials in $\theta^{\mu\nu}$, must satisfy the following system of equations

$$L^{(0)}\delta z^{(0)}=0,\quad L^{(0)}\delta z^{(l)}=f^{(l)}[a_i^{(m)},\phi^{(p)},\delta z^{(q)}],$$

where $L^{(0)}$ is the ordinary operator characterizing the ordinary zero modes:

$$(L^{(0)}\delta z)_i=\epsilon_{ijk}D_j\delta a_k\mp(D_i\delta\phi-i[\delta a_i,\phi])$$

and $f^{(l)}$ is a homogeneous polynomial of degree l in $\theta^{\mu\nu}$. The actual value of $f^{(l)}$ is immaterial to our argument, but for the fact that it does not depend on $\delta z^{(l)}$. Now, let us assume that there exists a solution to the previous set of equations; then, there are as many solutions as there are solutions to $L^{(0)}\delta z=0$. We know that, modulo gauge transformations that go to the identity at infinity, the number of linearly independent solutions to the ordinary zero mode equation is four.

3.2 Fundamental noncommutative BPS monopole configurations for SU(3). Two-monopole configurations

In this subsection the ordinary fields a_i and ϕ in eq. (3.1) will take values in the Lie algebra of SU(3) in the fundamental representation. Let us further assume that the asymptotic value of ϕ — and, hence, the asymptotic value of Φ, see eq. (2.6) — along the negative z-axis is given by

$$\phi(0,0,z\rightarrow-\infty)=v\,\vec{h}\cdot\vec{H},\tag{3.13}$$

where $\vec{H}=(H_1,H_2)$, H_1 and H_2 being the generators of the Cartan subalgebra of SU(3), and $\vec{h}=(h_1,h_2)$ is a unitary two-vector that unless otherwise stated will have non-vanishing scalar product with any root of SU(3).

For these boundary conditions the gauge SU(3) symmetry is broken down to U(1) $\times$ U(1). It is well known [25] that for this maximal breaking a solution to the ordinary BPS equations, $b_i=D_i\phi$, will have a magnetic vector $\vec{g}=n_1\frac{\vec{\beta}_1}{\beta_1^2}+n_2\frac{\vec{\beta}_2}{\beta_2^2}$, where the integers $n_1\geq 0$ and $n_2\geq 0$ are topological numbers and $\vec{\beta}_1$ and $\vec{\beta}_2$ are the unique set of simple roots of

SU(3) selected by the condition $\vec{h}\cdot\vec{\beta}_i > 0$. It is further well established [12] that these BPS solutions can be understood as multi-monopole configurations made out of two fundamental monopole solutions or their corresponding anti-monopoles. These fundamental monopole solutions have topological charges (n_1, n_2) equal to $(1,0)$ and $(0,1)$, respectively, and are obtained by embedding the SU(2) monopole in the SU(2) subgroups of SU(3) defined by the roots $\vec{\beta}_1$ and $\vec{\beta}_2$ of SU(3), respectively.

Let $T^a_{\beta_i}$, $a = 1,2,3$ and $i = 1,2$ be the generators of the SU(2) subgroup of SU(3) defined by the simple root $\vec{\beta}_i$. Then,

$$T^1_{\beta_i} = \frac{1}{\sqrt{2\beta_i^2}}\,(E_{\vec{\beta}_i} + E_{-\vec{\beta}_i}), \quad T^2_{\beta_i} = \frac{-i}{\sqrt{2\beta_i^2}}\,(E_{\vec{\beta}_i} - E_{-\vec{\beta}_i}), \quad T^3_{\beta_i} = \frac{1}{\beta_i^2}\,\vec{\beta}_i\cdot\vec{H},$$

where $E_{\vec{\beta}_i}$ is the generator of SU(3) defined by the root $\vec{\beta}_i$ in the Cartan-Weyl decomposition of the Lie algebra of SU(3): $[H_k, E_{\vec{\beta}_i}] = (\vec{\beta}_i)_k\, E_{\vec{\beta}_i}$. The fundamental monopoles with topological charges $(1,0)$ and $(0,1)$ are obtained by replacing i with 1 and 2, respectively, in the following equations

$$\begin{aligned} \phi^{(0)}_{\beta_i} &= \sum_{a=1,2,3} \phi^{(0)\,a}\, T^a_{\beta_i} + v\vec{h}\cdot\vec{H} - v\vec{h}\cdot\vec{\beta}_i\, T^3_{\beta_i} \\ a^{(0)}_{\beta_i} &= \sum_{a=1,2,3} a_i^{(0)\,a}\, T^a_{\beta_i}. \end{aligned} \tag{3.14}$$

$\phi^{(0)\,a}$ and $a_i^{(0)\,a}$ are the functions given in eq. (3.4) with the choice of positive sign for $H(r)$ and for $\lambda = v\,\vec{h}\cdot\vec{\beta}_i$. Of course, the previous field configurations are solutions to the noncommutative BPS equations $B_i = D_i\Phi$ at order zero in $h\theta^{\mu\nu}$.

Before computing the first-order-in-$\theta^{\mu\nu}$ corrections — $a_i^{(1)}$ and $\phi^{(1)}$ in eq. (3.1) — to the previous ordinary fundamental monopoles, we need some preparations. We shall choose the coordinate axis in the root space and the Cartan-Killing metric so that $\vec{\beta}_1 = (1,0)$ and $\vec{\beta}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$. The Gell-Mann generators of SU(3) will be denoted by $T^a = \frac{\lambda^a}{2}$, $a = 1\ldots 8$, where λ^a are the Gell-Mann matrices. Under the adjoint action of the SU(2) generators $T^a_{\beta_i}$, $a = 1,2,3$, the generators of SU(3) T^a, $a = 1\ldots 8$ can be sorted out into one triplet, two doublets and one singlet, which have the following value in terms of Gell-Mann matrices,

$$\begin{aligned} \beta_1 :\ &\text{Triplet} &&: \{T^1, T^2, T^3\} \\ &\text{Doublets} &&: \{T^4, T^5, T^6, T^7\}, \\ &\text{Singlet} &&: T^8, \\ \beta_2 :\ &\text{Triplet} &&: \{T^6, T^7, -\tfrac{1}{2}T^3 + \tfrac{\sqrt{3}}{2}T^8\}, \\ &\text{Doublets} &&: \{T^1, T^2, T^4, T^5\}, \\ &\text{Singlet} &&: -\tfrac{\sqrt{3}}{2}T^3 - \tfrac{1}{2}T^8. \end{aligned} \tag{3.15}$$

Denoting by T^s_β the singlet generator in the previous equation, it can be seen that the ordinary field configurations of eq. (3.14) can be written thus

$$\phi^{(0)} = \sum_{\text{triplet}} \phi^{t\,a} T^a_\beta + \phi^s T^s_\beta, \quad \phi^s = 2\,v\,\mathrm{Tr}\,(T^s_\beta\,\vec{h}\cdot\vec{H}). \tag{3.16}$$

$\phi^{t\,a}$, $a=1,2,3$, are given by the components of the ordinary SU(2) monopole. We are now ready to compute $a_i^{(1)}$ and $\phi^{(1)}$ in eq. (3.1) in the case at hand. Proceeding as in the SU(2) case — see eqs. (3.5) and (3.6) — , one obtains that $a_i^{(1)}$ and $\phi^{(1)}$ must satisfy the following equation:

$$(f_{ij}^{(1)} + F_{ij}^{(1,0)}) = \epsilon_{ijk}[(D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}]. \tag{3.17}$$

Now both sides of the equation take values in the enveloping algebra of SU(3) in the fundamental representation. Hence, eq. (3.17) is equivalent to the following two equations

$$\begin{aligned} a) \qquad \mathrm{Tr}\big[(f_{ij}^{(1)} + F_{ij}^{(1,0)})\big] &= \epsilon_{ijk}\mathrm{Tr}\big[(D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}\big], \\ b) \qquad \mathrm{Tr}\Big[\tfrac{\lambda^a}{2}(f_{ij}^{(1)} + F_{ij}^{(1,0)})\Big] &= \epsilon_{ijk}\mathrm{Tr}\Big[\tfrac{\lambda^a}{2}\big((D_k\phi)^{(1)} + \mathcal{O}_k^{(1,0)}\big)\Big]. \end{aligned} \tag{3.18}$$

As in the SU(2) case, the equality $a)$ in the previous equation only involves the zero order contributions to the field configurations: $a_i^{(0)}$ and $\phi^{(0)}$ in eqs. (3.14) and (3.16). Eq. $a)$ is thus a constraint on the parameters of the Seiberg-Witten map. Although $\phi^{(0)}$ has a non-vanishing component along the singlet generator $T^s_{\vec\beta}$, one may show that $a)$ in eq. (3.18) holds if, and only if, the parameters κ_2 and λ_1 of the Seiberg-Witten map in eq. (2.3) take the same values as in the SU(2) case — see eq. (3.10). Some computations lead to the conclusion that the equality $b)$ in eq. (3.18) is equivalent to the following equation

$$D_i^{(0)}(a'_j) - D_j^{(0)}(a'_i) = \epsilon_{ijk}\left(D_k^{(0)}\phi' - i[a'_k, \phi^{(0)}]\right), \tag{3.19}$$

where a'_j and ϕ' are defined in terms of $a_i^{(1)}$ and $\phi^{(1)}$ by the following identities:

$$\begin{aligned} a_i^{(1)} &= a'_i - \kappa_1\,\theta^{kl}\,D_i^{(0)}f_{kl}^{(0)} - i\kappa_3\,\theta_i{}^l\,[D_l^{(0)}\phi^{(0)}, \phi^{(0)}] - \kappa_4\,v\,\theta_i{}^j D_j^{(0)}\phi^{(0)} + \frac{\phi^s}{2\sqrt{3}}\theta_i{}^j\,D_j^{(0)}\phi^{(0)}, \\ \phi^{(1)} &= \phi' - i\lambda_2\,\theta^{ij}\,[f_{ij}^{(0)}, \phi^{(0)}] - \lambda_3\,v\,\theta^{ij}f_{ij}^{(0)} - \frac{\phi^s}{4\sqrt{3}}\,\theta^{ij}\,f_{ij}^{(0)}, \end{aligned} \tag{3.20}$$

respectively.

Eq. (3.19) is defining, modulo gauge transformations, the zero modes, or moduli, of the corresponding ordinary fundamental monopole. Hence, a'_i and ϕ' are given by appropriate linear combinations of the corresponding moduli with coefficients that depend linearly on $h\theta^{\mu\nu}$. This is completely analogous to what we found in the SU(2) case. However, we see that now $a_i^{(1)}$ and $\phi^{(1)}$ contain extra contributions, as compared with those in eq. (3.13), coming from the singlet part, $\phi^s\,T^s_{\vec\beta}$, of $\phi^{(0)}$. And yet, the complete noncommutative correction to the ordinary SU(3) BPS fundamental monopoles is a linear combination of ordinary zero modes and field redefinitions. Let us stress that the values of the real coefficients k_1, k_3, k_4, λ_2 and λ_3 that parametrize the ambiguity in the Seiberg-Witten map corresponding to field redefinitions have no bearing on the existence of noncommutative BPS solutions. However, the existence of these noncommutative field configurations demands $k_2 = -\frac{1}{2}$ and $\lambda_1 = \frac{1}{4}$, k_2 and λ_1 parametrizing the ambiguities of the Seiberg-Witten map that cannot be interpreted neither as field redefinitions nor as gauge transformations.

In ordinary space-time, there is another natural embedding of SU(2) into SU(3). This is the embedding along the remaining positive root $\vec\beta_3 = \vec\beta_1 + \vec\beta_2$. The embedding of the ordinary SU(2) monopole in the SU(2) subgroup of SU(3) defined by $\vec{\beta}_3$ has topological charges $(1,1)$ and is not a fundamental monopole but rather a two-monopole field configuration constituted by a fundamental monopole of type $(1,0)$ and another of type $(0,1)$ superimposed at the same point. The mass and magnetic charge of this $(1,1)$ two-monopole are the sum of those of its constituent fundamental monopoles — see [12] for further information. Obviously, the noncommutative counterpart of the previous ordinary two-monopole is given, at first order in $h\theta^{\mu\nu}$ and if eq. (3.10) holds, by eq. (3.20), but, now, λ is equal to $v\,\vec{h}\cdot\vec{\beta}_3$ and the generators of SU(3), $T^a_{\beta_3}$, are defined in terms the eight Gell-Mann matrices, λ^a, as follows

$$\begin{aligned} \text{Triplet} &: \{T^4, T^5, \frac{1}{2}T^3 + \frac{\sqrt{3}}{2}T^8\}, \\ \text{Doublets} &: \{T^1, T^2, T^6, T^7\}, \\ \text{Singlet} &: \frac{\sqrt{3}}{2}T^3 - \frac{1}{2}T^8. \end{aligned} \tag{3.21}$$

The labels Triplet, Doublets and Singlet refer to the behaviour of T^a, $a = 1\dots8$, under the adjoint action of the SU(2) generators $T^a_{\beta_3}$, $a = 1,2,3$.

The noncommutative field configuration we have just constructed has topological charges $(1,1)$ and mass M_3 equal to $M_1 + M_2$, with $M_1 = v\,\vec{h}\cdot\vec{\beta}_1$ and $M_2 = v\,\vec{h}\cdot\vec{\beta}_2$. Further, one may argue that, as is the case with its ordinary counterpart, there are eight zero modes, or moduli, associated with it. Indeed, the number of linearly independent normalizable zero modes can be obtained by computing the index of an operator that differs from the corresponding ordinary operator in ref. [12] by terms that are of order one in $h\theta^{\mu\nu}$. These terms are to be considered small continuous perturbations of the ordinary operator and hence they will not change the value of the index — this is actually what happens in the case of the chiral anomaly in ref. [26] and for fundamental monopoles. It is therefore natural to conclude that the noncommutative BPS configuration obtained for the root $\vec{\beta}_3$ is made out of two fundamental noncommutative monopoles: a β_1-monopole and a β_2-monopole.

Finally, it is straightforward to repeat the previous analysis for negatively charged monopoles, obtained as deformations of the embeddings of the SU(2) anti-monopole along the SU(2) subalgebras defined by the roots $\vec{\beta}_1$, $\vec{\beta}_2$ and $\vec{\beta}_3$. The same conclusions are reached as in the case of positively charged monopoles.

3.3 Noncommutative SO(5) theory and BPS massless monopoles

In ordinary space-time, when the unbroken gauge group is not the maximal torus of the broken gauge group, there show up massless monopoles [13]. These objects do not occur as isolated solutions to the equations of motion, but manifest themselves in multi-monopole field configurations as clouds surrounding massive monopoles and carrying non-abelian magnetic charges. The simplest example where these field configurations with massless monopoles occur is furnished by SO(5) Yang-Mills-Higgs theory, with SO(5) broken down to SU(2) $\times$ U(1). An eight-moduli family of BPS solutions was found in ref. [27] that contains one fundamental massive β-monopole and one massless γ-monopole. The Higgs field of this configuration satisfies the boundary condition $\phi(0,0,z\rightarrow-\infty)=v\vec{h}\cdot\vec{H}$, with $\vec{h}\cdot\vec{\beta}>0$ and $\vec{h}\cdot\vec{\gamma}=0$. $\{\vec{\beta},\quad\vec{\gamma}\}$ is a set of simple roots of SO(5). We label the roots of SO(5) as follows: $\{\pm\vec{\alpha},\pm\vec{\beta},\pm\vec{\gamma},\pm\vec{\mu}\}$, where

$$\vec{\alpha}=(0,1)\quad\vec{\beta}=\Big(-\frac{1}{2},\frac{1}{2}\Big)\quad\vec{\gamma}=(1,0)\quad\vec{\mu}=\Big(\frac{1}{2},\frac{1}{2}\Big).$$

To display the BPS two-monopole solution in question some notation is needed. Let $E_{\pm\delta}$ be the rising and lowering generators of SO(5) defined by the root $\vec{\delta}$ of the latter. Let ${T^a}_\delta$ denote, for $a=1,2,3$, the generators of the SU(2) subgroup of SO(5) defined by the root $\vec{\delta}$. Then, any element, Q, of the Lie algebra of SO(5) admits the following decomposition:

$$Q=\sum_{a=1}^{3}Q(1)^a{T_\alpha}^a+\sum_{a=1}^{3}Q(2)^a{T_\gamma}^a+\mathrm{tr}Q(3)M,\;\;M=\frac{i}{\sqrt{\beta^2}}\begin{pmatrix}E_\beta & -E_{-\mu}\\ E_\mu & E_{-\beta}\end{pmatrix},$$

where $Q(3)^*=-\sigma_2Q(3)\sigma_2$, with σ_2 denoting the second Pauli matrix. Then, the field configuration constituted by a massive $\vec{\beta}$-monopole and a massless $\vec{\gamma}$-monopole has the following components: $Q(s)^a=a_i(s)^a$ or $\phi(s)^a$, $s=1,2$ and $a=1,2,3$, and $Q(3)=a_i(3)$ or $\phi(3)$, with

$$\begin{aligned}
a_i(1)^a&=\epsilon_{aim}\,A(r)\tfrac{x_m}{r}, & \phi(1)^a&=H(r)\,\tfrac{x_a}{r},\\
a_i(2)^a&=\epsilon_{aim}\,G(r,b)\tfrac{x_m}{r}, & \phi(2)^a&=G(r,b)\,\tfrac{x_a}{r},\\
a_i(3)&=\sigma_i\,F(r,b), & \phi(3)&=-iIF(r,b),\\
A(r)&=\tfrac{1}{r}-\tfrac{v}{\sinh(vr)}, & H(r)&=\tfrac{1}{r}-v\coth(vr),\\
F(r,b)&=\tfrac{v}{\sqrt{8}\cosh(vr/2)}\,L(r,b)^{1/2}, & G(r,b)&=A(r)L(r,b),\\
L(r,b)&=\big[1+\tfrac{r}{b}\,\coth(\tfrac{vr}{2})\big]^{-1}, & & b>0.
\end{aligned}\tag{3.22}$$

σ_i and I stand for the Pauli matrices and the 2×2 identity matrix, respectively, and $v=2\vec{\beta}\cdot\vec{h}$. Notice that under the unbroken SU(2) subgroup furnished by $\vec{\gamma}$, $Q^a(1)$, $Q^a(2)$ and $Q(3)$ transform as three singlets, a triplet and a complex doublet. The SU(2) triplet $Q^a(2)$ decays as $1/r$ in the region $1/v\lesssim r\lesssim b$. This is the non-abelian cloud representing classically the massless monopole which is charged under the unbroken SU(2) — for further discussion, see refs. [13, 4].

The purpose of this subsection is to see whether, at first order in $h\theta^{\mu\nu}$, there exist solutions to the noncommutative BPS equation, $B_i-D_i\Phi=0$, that are formal power series in $h\theta^{\mu\nu}$ and that go to the field configuration in eq. (3.22) as $h\theta^{\mu\nu}\rightarrow0$. We shall assume that the generators of SO(5) are in the fundamental representation. The contribution, at first order in $h\theta^{\mu\nu}$, to the non-abelian BPS equation reads

$$E\equiv f_{ij}^{(1)}+F_{ij}^{(1,0)}-\epsilon_{ijk}[(D_k\phi)^{(1)}+\mathcal{O}_k^{(1,0)}]=0.$$

The notation is the same as in subsection 2.1, but now E belongs to the enveloping algebra of SO(5) in the fundamental representation. In the previous cases, since we were dealing with SU(N) groups in the fundamental representation, any element of the enveloping algebra could be expressed as a linear combination of the generators of the Lie algebra and the identity; this is no longer the case now. The generators of the Lie algebra of SO(5) in the fundamental representation can be taken as pure imaginary hermitian — and therefore antisymmetric — matrices; then, the enveloping algebra includes also all the real symmetric matrices. It is possible to construct a basis $\{R^a\}$ of the enveloping algebra of SO(5) in the fundamental representation that is made out of the generators of SO(5), $\{T^a\}$, $a = 1 \ldots 10$ and a basis $\{S^a\}$, $a = 1 \ldots 15$ of the real symmetric matrices. The whole basis can be made orthogonal with respect to the trace operation: $\mathrm{Tr}\, R^a R^b \propto \delta^{ab}$. Using this orthogonal basis, the previous equation can be projected out onto a given element of the former just by first multiplying the latter by the element in question and, then, taking traces:

$$E = 0 \quad \Leftrightarrow \quad \mathrm{Tr}\, S^a\, E = 0,\ \forall a = 1 \ldots 15 \quad \text{and} \quad \mathrm{Tr}\, T^a\, E = 0,\ \forall a = 1 \ldots 10.$$

Since the trace of an antisymmetric matrix times a symmetric one vanishes, it turns out that $f_{ij}^{(1)}$ and $(D_k\phi)^{(1)}$ drop out from $\mathrm{Tr}\, S^a\, E = 0$. Hence, only the ordinary field configuration enters the equations $\mathrm{Tr}\, S^a\, E = 0$, which are thus turned into the following constraint on the parameters of the Seiberg-Witten map:

$$\begin{aligned}\mathrm{Tr}\, S^a[F_{ij}^{(1,0)\,\mathrm{st}} - \epsilon_{ijk}\mathcal{O}_k^{(1,0)\,\mathrm{st}}] &= -\mathrm{Tr} S^a[D_i^{(0)}(\,\kappa_2\,\theta_j{}^k\{(D_k\phi)^{(0)},\phi^{(0)}\}) - (i \leftrightarrow j)] \\ &\quad + \epsilon_{ijm}\mathrm{Tr}\, S^a\, D_m^{(0)}\,[\,\lambda_1\,\theta^{kl}\{f_{kl}^{(0)},\phi^{(0)}\}]. \end{aligned} \tag{3.23}$$

The reader is referred to subsection 3.2 for notation. The superscript "st" shows that the corresponding object is computed by using the standard form — all free parameters set to zero — of the Seiberg-Witten map. Now, substituting eq. (3.22) in eq. (3.23), one ends up with the conclusion that the resulting equation holds if, and only if, κ_2 and λ_1 take the values quoted in eq. (3.10).

It remains to solve $\mathrm{Tr}\, T^a\, E = 0$. Since $\{T^a, T^b\}$ is a symmetric matrix, the previous equation boils down to

$$\begin{aligned} D_i^{(0)}(a_j') - D_j^{(0)}(a_i') &= \pm\epsilon_{ijk}\left(D_k^{(0)}\phi' - i[a_k',\phi^{(0)}]\right) \\ a_i' &= a_i^{(1)} + \kappa_1\,\theta^{kl}\,D_i^{(0)} f_{kl}^{(0)} + i\kappa_3\,\theta_i{}^l\,[(D_l\phi)^{(0)},\phi^{(0)}] + \kappa_4\,v\theta_i{}^j D_j^{(0)}\phi^{(0)}, \\ \phi' &= \phi^{(1)} + i\lambda_2\,\theta^{kl}\,[f_{kl}^{(0)},\phi^{(0)}] + \lambda_3\,v\,\theta^{ij} f_{ij}^{(0)}, \end{aligned} \tag{3.24}$$

where $(a_i^{(0)},\phi^{(0)})$ denotes the ordinary field configuration of eq. (3.22). Hence, the first-order-in-$h\theta^{\mu\nu}$ BPS corrections to the ordinary field configuration are given, again, by the terms of the Seiberg-Witten map associated to field redefinitions plus θ-dependent linear combinations of the ordinary zero modes.

One may care to compute the first-order-in-$h\theta^{\mu\nu}$ BPS corrections to ordinary fundamental monopoles for SO(5) in the fundamental representation. Proceeding as in the previous paragraphs one concludes that they exist if, and only if, eq. (3.10) holds, and that they are given by eq. (3.24), if $(a_i^{(0)},\phi^{(0)})$ denotes now the ordinary fundamental monopoles. Let us stress that we have shown that, for SU(2) and SO(5) in their fundamental repere-sentation, $a^{(1)},\phi^{(1)}$ are given by the same type of corrections. A result that has its origin partially in the fact that for both groups $\mathrm{Tr}\, T^a\{T^b, T^c\} = 0$. Notice that $\mathrm{Tr}\, T^a\{T^b, T^c\} \neq 0$ for SU(3).

4. Static solutions to the BPS Yang-Mills-Higgs equations at first order in $h\theta^{\mu\nu}$

In the previous section, we have seen that for some gauge simple groups only if the parameters labeling the ambiguity of the Seiberg-Witten map are appropriately chosen there exist noncommutative BPS (multi-)monopoles that are power series in $h\theta^{\mu\nu}$ and that go to a given ordinary BPS (multi-)monopole configuration as $h\theta^{\mu\nu} \to 0$. The next question to ask is whether given an ordinary BPS (multi-)monopole configuration there exists for any value of κ_2 and λ_1 a solution to the noncommutative Yang-Mills-Higgs equations in the BPS limit that has the following properties: it is static, it is a power series in $h\theta^{\mu\nu}$ and it goes to the given ordinary BPS (multi-)monopole configuration as $h\theta^{\mu\nu} \to 0$. Notice that the noncommutative BPS equations had contributions that were proportional to the identity matrix, and this was part of the problem, whereas in the noncommutative Yang-Mills-Higgs equations in the BPS limit for simple groups, which are displayed in eq. (2.11), no contribution of that sort occurs.

4.1 SU(2) case

At zero order in $h\theta^{\mu\nu}$, the equations of motion are the ordinary ones and hence they are satisfied by $a_i^{(0)}, \phi^{(0)}$ — we use the notation of eq. (3.1) — given by ordinary BPS (multi-)monopole configurations. Let us choose the gauge $a_0 = 0$. After carrying out some simplifications, it can be shown that the contributions to eqs. (2.11), at first order in $h\theta^{\mu\nu}$ and for time independent field configurations, read

$$\begin{aligned} D_i D_i \phi' - i D_i [a'_i, \phi] - i[a'_i, D_i \phi] &= 0, \\ D_i(D_i a'_j - D_j a'_i) - i[a'_i, f_{ij}] + i[\phi', D_j\phi] + i[\phi, D_j\phi' - i[a'_j, \phi]] &= 0, \end{aligned} \tag{4.1}$$

$$\begin{aligned} a'_j &= a_j^{(1)} + \kappa_1\, \theta^{kl}\, D_j f_{kl} + i\kappa_3\, \theta_j{}^l\, [(D_l\phi), \phi] + v\kappa_4\, \theta_j{}^l D_l \phi \\ \phi' &= \phi^{(1)} + i\lambda_2\, \theta^{kl}\, [f_{kl}, \phi] + v\, \lambda_3\, \theta^{ij} f_{ij} \end{aligned}$$

where $D_i = D_i^{(0)} = \partial_i - i[a_i^{(0)},\quad]$, $f_{ij} = f_{ij}^{(0)}$, $a_i = a_i^{(0)}$ and $\phi = \phi^{(0)}$, $a_i^{(0)}$ and $\phi^{(0)}$ being the fields defining the ordinary BPS SU(2) (anti-)monopole. It is natural to look for a'_i and ϕ' such that

$$a'_i(\vec{x}) \sim \frac{1}{|\vec{x}|^2} \quad \text{and} \quad \phi'(\vec{x}) \sim \frac{1}{|\vec{x}|^2} \quad \text{as} \quad |\vec{x}| \to \infty. \tag{4.2}$$

Note that one readily deduces from eq. (2.7) that the terms that go with κ_1, κ_2 and κ_3 and λ_2 and λ_3 in eq. (4.1) satisfy the previous boundary conditions and that these boundary conditions guarantee that there will be no $\theta^{\mu\nu}$ dependent contributions to the magnetic charge defined in eq. (2.8). The latter contributions would put into jeopardy the interpretation of the magnetic charge as a topological quantity.

Let us analyse the equations for ϕ' and a'_i. Using the fact that $a^{(0)}$ and $\phi^{(0)}$ satisfy the ordinary BPS equations, the first equation in eq. (4.1) leads to

$$D_i(D_i\phi' - i[a'_i, \phi] \mp \epsilon_{ijk}D_j a'_k) = 0. \tag{4.3}$$

Introducing the four-vector fields in three dimensions $\bar{a}'_\mu = (a'_i, \phi')$ and $\bar{a}_\mu = (a_i^{(0)}, \phi^{(0)})$, one may cast eq. (4.3) into the form

$$\bar{D}_\mu(\bar{D}_\mu \bar{a}'_4 - \bar{D}_4 \bar{a}'_\mu \mp \epsilon_{\mu 4\rho\sigma}\bar{D}_\rho \bar{a}'_\sigma) = 0.$$

This equation is of the type $\bar{D}_\mu \bar{X}_{\mu 4} = 0$ with $\bar{D}$ in the background of a self-dual field $\bar{a}_\mu$ and with $\bar{X}_{\mu 4}$ being self-dual. Using the techniques in ref. [21], one may show that the only normalizable solutions to this equation are those satisfying $X_{i4} = 0$. Notice that $\bar{D}_\mu \bar{a}'_4 - \bar{D}_4 \bar{a}'_\mu \mp \epsilon_{\mu 4\rho\sigma}\bar{D}_\rho \bar{a}'_\sigma$ must be a smooth function of $\vec{x}$ such that it decays as $1/|\vec{x}|^2$ as $|\vec{x}| \to \infty$ and, hence, normalizable in three dimensions. Now, $\bar{X}_{\mu 4} = 0$ yields

$$D_i\phi' - i[a'_i, \phi] \mp \epsilon_{ijk}D_j a'_k = 0. \tag{4.4}$$

This is precisely the equation of the zero modes in the background of an ordinary BPS SU(2) (anti-)monopole. Going back to the second equation in eq. (4.1), using the result in eq. (4.4) and the condition $f_{ij} = \pm\epsilon_{ijk}D_k\phi$, we arrive at

$$D_i(D_i a'_j - D_j a'_i \mp i[\phi, \epsilon_{ijl}a'_l] \mp \epsilon_{ijl}D_l\phi') = 0,$$

which is automatically satisfied if eq. (4.4) holds. We therefore conclude that ϕ', a'_i which satisfy the boundary conditions in eq. (4.2) — see comments below eq. (4.2) — are just linear combinations of the zero modes of the ordinary BPS (anti-)monopole with $\theta^{\mu\nu}-$dependent coefficients. We thus conclude that there are solutions, for SU(2) and at first order in $h\theta^{\mu\nu}$, to the noncommutative Yang-Mills-Higgs equations in the BPS limit, whatever the value of the parameters labeling the ambiguity of the Seiberg-Witten map. These solutions are given by the field redefinitions of the ordinary BPS (anti-)monopole furnished by the Seiberg-Witten map plus some appropriate linear combinations of the ordinary SU(2) zero modes.

4.2 SU(3) case

Let $(a_i^{(0)},\ \phi^{(0)})$ denote the ordinary BPS monopole and two-monopole solutions in eq. (3.14). Let $D_i = \partial_i - i[a_i^{(0)},\quad]$, $f_{ij} = \partial_i a_j^{(0)} - \partial_j a_i^{(0)} - i[a_i^{(0)}, a_j^{(0)}]$, $\phi = \phi^{(0)}$, and let a'_j and ϕ' be given by

$$\begin{aligned} a'_k &= a_k^{(1)} + \kappa_1\,\theta^{ij}\,D_k f_{ij} + \frac{\kappa_2\phi^s}{\sqrt{3}}\,\theta_k{}^j\,D_j\phi + \frac{\kappa_2}{\sqrt{3}}\theta_k{}^j\,(D_j\phi)^a\phi^{ta}T^s_\beta + i\kappa_3\,\theta_k{}^l\,[(D_l\phi),\phi] \\ &\quad +\kappa_4\,v\,\theta_k{}^j D_j\phi \\ \phi' &= \phi^{(1)} + \frac{\lambda_1\phi^s}{\sqrt{3}}\theta^{ij}\,f_{ij} + \frac{\lambda_1}{\sqrt{3}}\,\theta^{ij} f^a_{ij}\,\phi^{ta}T^s_\beta + i\lambda_2\,\theta^{ij}\,[f_{ij},\phi] + \lambda_3\,v\,\theta^{ij}\,f_{ij}. \end{aligned} \tag{4.5}$$

See subsection 3.2 for notation. Then, for SU(3), the first order in $h\theta^{\mu\nu}$ contribution to the noncommutative Yang-Mills-Higgs equations of eq. (2.11) in the gauge $a_0 = 0$ and for time independent field configurations reads

$$
\begin{aligned}
&\mathrm{Tr}\, T^a \big[2D_jD_j\phi' - 2iD_j[a'_j, \phi] - 2i[a'_j, D_j\phi]\big] = \\
&\qquad -\mathrm{Tr}\, T^a\theta^{ij}\bigg[-\frac{1}{2}D_m\{D_m\phi, f_{ij}\} - D_j\{D_m\phi, f_{mi}\} - D_m\{D_j\phi, f_{mi}\}\bigg], \\
&\mathrm{Tr}\, T^a\big[-2D_i(D_i\, a'_k - D_k\, a'_i) + 2i[a'_i, f_{ik}] - 2i[\phi', D_k\phi] - 2i[\phi, -i[a'_k, \phi] + D_k\, \phi']\big] = \\
&\qquad -\mathrm{Tr}\, T^a\bigg[\theta^i{}_k\bigg(-\frac{1}{4}D_i\{f_{mn}, f_{mn}\} - D_m\{f_{ni}, f_{mn}\} \\
&\qquad\qquad -\frac{1}{2}D_i\{D_m\phi, D_m\phi\} + D_m\{D_i\phi, D_m\phi\}\bigg) \\
&\qquad\qquad +\theta^{ij}\bigg(\frac{1}{2}D_m\{f_{mk}, f_{ij}\} + D_i\{f_{mj}, f_{km}\} \\
&\qquad\qquad -D_m\{f_{mi}, f_{kj}\} - D_i\{D_j\phi, D_k\phi\}\bigg)\bigg].
\end{aligned} \tag{4.6}
$$

The non-zero traces that occur on the r.h.s of both equalities in eq.(4.6) are of the type $\mathrm{Tr}\, T^a\{T^b_\beta, T^c_\beta\}$. Since $\{T^b_\beta, T^c_\beta\}$ behaves as a singlet under the SU(2) Lie algebra generated by $\{T^c_\beta\}_{c=1,2,3}$, the r.h.s. is only nonzero if T^a is the SU(2) singlet generator. The corresponding l.h.s of the equations will pick up only the components of a'_i, ϕ' along the singlet, since for the basis in eqs. (3.15) and eq. (3.22), $\mathrm{Tr}\, T^aT^b = \frac{1}{2}\delta^{ab}$ holds and the SU(2) subalgebra defined by the root $\vec{\beta}$ acts irreducibly on the specified representations. Hence, the equations for the components of ϕ' and a'_k along the singlet decouple from the rest. We can express ϕ' and a'_i as follows: $\phi' = \phi'^s + \phi''$, with ϕ'^s being the component along the SU(2) singlet, and analogously $a'_i = a'^s_i + a''_i$.

Let us first analyse the equations for ϕ'' and a''_i. In this case the r.h.s. of the equalities in eq. (4.6) vanishes, so that we are left with the same equations given by the first two lines of eq. (4.1), whose solutions for the boundary conditions of eq. (4.2) are given by the zero modes in the background of the ordinary SU(3) BPS (two-)monopole.

It remains to solve the equations for the components, ϕ'^s, a'^s_i, along the singlet. For each SU(2) embedding in section 3.2, we choose the corresponding basis in eq. (3.15) and eq. (3.22) and take T^a in eq. (4.6) to be the corresponding singlet generator. Now, for each basis the property $\mathrm{Tr}\, T^s_\beta\{T^a_\beta, T^b_\beta\} = \frac{1}{2\sqrt{3}}\delta^{ab}$ holds, so that we end up with the following equations:

$$
\begin{aligned}
\partial_i\partial_i\phi'^s &= \frac{1}{2\sqrt{3}}\,\theta^{ij}\left[\frac{1}{2}\partial_k[(D_k\phi)^a(f_{ij})^a] + \partial_j[(D_k\phi)^a(f_{ki})^a] + \partial_k[(D_j\phi)^a(f_{ki})^a]\right], \\
\partial_i\partial_i a'^s_j - \partial_j\partial_i a'^s_i &= \frac{1}{2\sqrt{3}}\theta^i{}_j\Big[-\frac{1}{4}\,\partial_i[f^a_{mn}f^a_{mn}] - \partial_m[f^a_{ni}f^a_{mn}] - \frac{1}{2}\,\partial_i[(D_m\phi)^a(D_m\phi)^a] \\
&\quad +\partial_m[(D_i\phi)^a(D_m\phi)^a]\Big] + \frac{1}{2\sqrt{3}}\,\theta^{ik}\Big[\frac{1}{2}\partial_m[f^a_{mj}f^a_{ik}]\Big] + \partial_i[f^a_{mk}f^a_{jm}] \\
&\quad -\partial_m[f^a_{mi}f^a_{jk}] - \partial_i[(D_k\phi)^a(D_j\phi)^a]\Big].
\end{aligned}
$$

The computation of the r.h.s. of both equations for the field configurations in eqs. (3.14) yields

$$\begin{aligned}
\partial_i\partial_i\phi'^s &= \theta^{ij}\epsilon_{ijk}\,x^k f(r),\\
\partial_i\partial_i a'^s_j - \partial_j\partial_i a'^s_i &= 2\theta^i{}_j x^i f(r),\\
f(r) &= \frac{1}{2\sqrt{3}}\left[\frac{1}{2r}\frac{d}{dr}H'^2 + \frac{1}{r}\frac{d}{dr}\left(\frac{K'}{r}\right)^2\right].
\end{aligned}$$

The general solution to each of these equations is the sum of a particular solution plus a solution to the homogeneous equation. The homogeneous equation for ϕ'^s has no smooth solution that vanishes at infinity, while the homogeneous equation for a'^s_i has as non-singular solutions total derivatives which are equivalent to gauge transformations. Therefore we just need to find non-singular particular solutions that respect the boundary conditions. Choosing the following ansätze,

$$\phi'^s = \theta^{ij}\,\epsilon_{ijk}x^k g(r), \quad a'^s_j = \theta^i{}_j x^i h(r), \tag{4.7}$$

one finds the following solution

$$\begin{aligned}
g(r) = \frac{1}{2}\,h(r) &= \frac{1}{4\sqrt{3}}\,\frac{H(1-K)(3-K)}{r^3} \\
&= -\frac{1}{16\sqrt{3}r^4}\,\mathrm{csch}^3(r\lambda)[r\lambda\cosh(r\lambda)(1+4(r\lambda)^2) \\
&\quad -r\lambda\cosh(3r\lambda) + 2\sinh(r\lambda)(-1-2(r\lambda)^2+\cosh(2r\lambda))],
\end{aligned} \tag{4.8}$$

where $\lambda = v\,\vec{h}\cdot\vec{\beta}$. Putting it all together and realizing that the singlet contributions to a'_i and ϕ' in eq. (4.1) are proportional to the previously given $g(r)$, one ends up with the following family of static solutions to the first order in $h\theta^{\mu\nu}$ equations of motion:

$$\begin{aligned}
\phi^{(1)} &= \delta\phi^{(0)} + (1-4\lambda_1)\theta^{ij}\epsilon_{ijk}x^k g(r)T^s_\beta - \frac{\lambda_1\,\phi^s}{\sqrt{3}}\theta^{ij}\,f_{ij} - i\lambda_2\,\theta^{ij}\,[f_{ij},\phi] - \lambda_3\,v\,\theta^{ij}\,f_{ij}\\
a^{(1)}_i &= \delta a^{(0)}_i + (4\kappa_2+2)\theta^j{}_i x^j g(r)T^s_\beta - \kappa_1\,\theta^{kl}\,D_i f_{kl} - \frac{\kappa_2\phi^s}{\sqrt{3}}\,\theta_i{}^j D_j\phi\\
&\quad -i\kappa_3\theta_i{}^j[D_j\phi,\phi] - \kappa_4\,v\,\theta_i{}^j\,D_j\phi.
\end{aligned} \tag{4.9}$$

$\delta\phi^{(0)}$ and $\delta a^{(0)}_i$ denote any linear combination of the zero modes of the corresponding ordinary BPS configuration with coefficients that depend linearly on $h\theta^{\mu\nu}$. D_i, f_{ij} and ϕ have been defined at the beginning of this subsection. The solutions reported in eq. (4.9) are well behaved at $r = 0$ and the behaviour at infinity is such that the noncommutative corrections respect the ordinary boundary conditions and do not contribute to the magnetic charge. When $\kappa_2 = -\frac{1}{2}$, $\lambda_1 = \frac{1}{4}$, values for which there exist solutions to the noncommutative BPS equations, the singlet contributions vanish and we recover the field configurations that solve the noncommutative BPS equations for SU(3) — see eq. (3.20).

The solution in eq. (4.9), which exists for any value of the parameters of the Seiberg-Witten map defined in eq. (2.3) — with $w = 0$, of course — , constitutes a noncommutative deformation of the ordinary BPS field configuration obtained by embedding the ordinary BPS SU(2) monopole along the root $\vec{\beta}$, with $\vec{\beta}=\vec{\beta}_i$, $i=1,2,3$. The mass, M_β, of the complete static field configuration, which in general is a noncommutative non-BPS field configuration, is obtained by substituting $a_i = a^{(0)} + h a_i^{(1)}$ and $\phi = \phi^{(0)} + h\phi^{(1)}$ in eq. (2.9). After a little algebra one ends up with a number of integrals that have to be worked out numerically. The final answer for M_β is then given by

$$M_\beta = M_{\text{ordinary}} + 0.10274\ h^2\theta^{ij}\theta^{ij}\,\lambda^5\left[\left(\kappa_2+\frac{1}{2}\right)^2 + 2\left(\lambda_1-\frac{1}{4}\right)^2\right] + O(h^3\theta^3). \qquad (4.10)$$

$M_{\text{ordinary}} = 4\pi\lambda$ is the ordinary mass and $\lambda = v\,\vec{\beta}\cdot\vec{h}$, with $\vec{h}$ defined in eq. (3.13) and such that $\vec{\beta}_i\cdot\vec{h}>0$, $\forall i$. Recall that h^2 does not denote $\vec{h}\cdot\vec{h}$. Notice that the quadratic contributions in $h\theta^{\mu\nu}$ to M_β are not affected by the quadratic contributions in $h\theta^{\mu\nu}$ to field configurations since $(a_i^{(0)},\phi^{(0)})$ satisfies the ordinary BPS equations.

Let $\vec{\beta}_1=(1,0)$ and $\vec{\beta}_2=(-\frac{1}{2},\frac{\sqrt{3}}{2})$; then, $\vec{h}$ is given by $\vec{h}=(\omega,\sqrt{1-\omega^2})$, $0<\omega<\frac{\sqrt{3}}{2}$, for $\vec{\beta}_i\cdot\vec{h}>0$, $i=1,2$.

Now, if $\vec{\beta}=\vec{\beta}_1$, then, eq. (4.9) corresponds generically to a noncommutative non-BPS monopole with topological vector charge $(1,0)$. If $\vec{\beta}=\vec{\beta}_2$, then eq. (4.9) corresponds generically to a noncommutative non-BPS monopole with topological vector charge $(0,1)$. Finally, if $\vec{\beta}=\vec{\beta}_3=\vec{\beta}_1+\vec{\beta}_2$ we have generically a noncommutative non-BPS two-monopole configuration with topological charges $(1,1)$. We do not think that — unless $\kappa_2=-\frac{1}{2}$, $\lambda_1=\frac{1}{4}$ holds — the two-monopole configuration is stable. Indeed a little algebra reveals that $M_{\beta_3}>M_{\beta_1}+M_{\beta_2}$, if $\kappa_2=-\frac{1}{2}$, $\lambda_1=\frac{1}{4}$ is not satisfied and $0<\omega<\frac{\sqrt{3}}{2}$. Indeed,

$$M_{\beta_3}-(M_{\beta_1}+M_{\beta_2})=0.10274\,h^2\theta^{ij}\theta^{ij}v^5\left[\left(\kappa_2+\frac{1}{2}\right)^2+2\left(\lambda_1-\frac{1}{4}\right)^2\right]\left[\frac{15}{16}\omega(3-4\,\omega^2)\right]>0.$$

This inequality suggests that the noncommutative character of space gives rise to a repulsive interaction between the $\vec{\beta}_1$-monopole and the $\vec{\beta}_2$-monopole that constitute the field configuration for $\vec{\beta}_3$. Hence, an infinitesimal disturbance of the static configuration with mass M_{β_3} will make the object decay into a system constituted by two infinitely separated noncommutative non-BPS monopoles, one of type $\vec{\beta}_1$ and the other of type $\vec{\beta}_2$. Notice that the latter two-monopole system has mass equal to $M_{\beta_1}+M_{\beta_2}$ and belongs to the topological class of the non-commutative non-BPS $\vec{\beta}_3$-field configuration. This result casts doubts on the stability of other non-BPS multi-monopole configurations for simple gauge groups. The extension to the case of negatively charged monopoles is once again trivial; anti-monopole configurations pick up a minus sign in the term proportional to $(1-4\lambda_1)$ in eq. (4.9) and their energy is equal to that of their positively charged partners.

4.3 SO(5) case

As in the SU(2) case — but not for SU(3) — , the traces of the type $\mathrm{Tr}\,T^a\{T^b,T^c\}$ vanish. We are thus left precisely with the equations that one finds in eq. (4.1). Repeating the analysis made below eq. (4.1), we arrive at the same result, i.e., that the first order in $h\theta^{\mu\nu}$ deformations of the ordinary field configurations are given by the field redefinitions determined by the Seiberg-Witten map plus solutions to the zero mode equations in the background of the ordinary monopole.

5. Summary, conclusions and outlook

For three specific gauge groups — SU(2), SU(3) and SO(5) — in their fundamental representations, we have discussed the existence of monopole and some two-monopole field configurations in noncommutative Yang-Mills-Higgs theories in the BPS limit. We have looked for field configurations that are formal power series in $h\theta^{\mu\nu}$ and worked at first order in $h\theta^{\mu\nu}$. We have considered a commutative time and the most general Seiberg-Witten map that leads to an action that, in the gauge $a_0 = 0$, contains only first order time derivatives of the fields and is a quadratic functional of them. We have shown that there is no monopole solution to these BPS equations unless two a priori free parameters of the Seiberg-Witten map are tuned to two concrete values — see eq. (2.3) and (3.10). These free parameters are those free parameters that are not related with field redefinitions nor with gauge transformations. The same state of affairs was met when studying the two-monopole solution that in the limit $h\theta^{\mu\nu} \to 0$ goes to the ordinary β_3-two-monopole solution of SU(3) and the noncommutative field configuration that in that very limit yields the ordinary one-massive-one-massless two-monopole solution of SO(5). We then showed that whatever the values of the parameters of our Seiberg-Witten map the noncommutative Yang-Mills-Higgs equations admit, at first order in $h\theta^{\mu\nu}$, monopole field configurations that solve them and have the same magnetic charge — although for SU(3) they have different mass — as the ordinary monopoles they go to in the limit $h\theta^{\mu\nu} \to 0$. For SU(2) and SO(5) the first order in $h\theta^{\mu\nu}$ corrections correspond to field redefinitions of the corresponding ordinary object. This is not so for SU(3). In this case the masses of the field configurations have contributions that depend quadratically on $\theta^{\mu\nu}$, so that the mass of the static $\vec{\beta}_3 = \vec{\beta}_1 + \vec{\beta}_2$ field configuration is larger than the sum of the masses of its constituents: the $\vec{\beta}_1$-monopole and the $\vec{\beta}_2$-monopole, $\vec{\beta}_1$ and $\vec{\beta}_2$ being given simple roots of SU(3). This static $\vec{\beta}_3$ non-BPS field configuration seems to be unstable. Let us now state the main conclusions of this paper. First, at first order in $h\theta^{\mu\nu}$, there are BPS monopole solutions in noncommutative SU(2), SU(3) and SO(5) Yang-Mills-Higgs theory provided the Seiberg-Witten map is appropriately chosen. This is in sharp contrast with the instanton case, where no solutions to the noncommutative self-duality equations could be found already at first order in $h\theta^{\mu\nu}$ — see ref. [21]. Second, the parameters κ_2 and λ_1 of the Seiberg-Witten map in eq. (2.3) have physics in them. Indeed, the properties of the moduli space of the Yang-Mills-Higgs equations depend on their values: if they take the values of eq. (3.10), the elements of the moduli space are BPS objects, and if they do not, they are non-BPS elements. Notice that the masses of generics non-BPS SU(3) monopoles depend on κ_2 and λ_1, see eq. (4.10). For simple gauge groups, the fact that the value of the parameters labeling the ambiguity in the Seiberg-Witten map which is not related to field redefinitions nor to gauge transformations may have physical consequences is an issue which cannot be overlooked when considering the phenomenological applications of the noncommutative theories constructed within the formalism of refs. [14, 15]. Third, for generic values of κ_2 and λ_1 noncommutative multi-monopole solutions may become unstable even if they deform ordinary BPS multi-monopole configurations. There are many directions in which the piece of research presented in this paper can be further developed. We shall mention just a few of them. First, the computation of the corrections at second order in $h\theta^{\mu\nu}$ to the (multi-)monopole field configurations worked out here. We show in the appendix that, at variance with the case of instantons — see ref. [21], Derrick's theorem poses no obstruction on the existence — for $a_0 = 0$ — of static field configurations that solve the equations of motion at second order in $h\theta^{\mu\nu}$. Second, it will be interesting to consider other representations and other gauge groups. Notice that the field equations take values in the enveloping algebra of the gauge group, so choosing a representation may have physical consequences. Third, it is very much needed to analyse the question of the stability of non-BPS multi-monopole configurations, such as the configuration of eq. (4.9), by using the methods of ref. [29]. Finally, it is a pressing need to construct supersymmetric generalizations of the noncommutative models presented here. In ordinary space-time, BPS monopoles unavoidably occur in some of these theories, so one wonders whether extended supersymmetry has any bearing on the value of the parameters of the Seiberg-Witten map and, in particular, if the values for κ_2 and λ_1 in eq. (3.10) are dictated by supersymmetry.

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A. Solutions at higher order in θ and Derrick's theorem

In ref. [21], after obtaining the most general solution to the noncommutative equations of motion for the first-order-in-$\theta^{\mu\nu}$ deformations of the BPST instanton in noncommutative SU(3) Yang-Mills theory, it was shown by studying the behaviour of the action under dilatations up to order $h^2\theta^2$ — i.e., by using Derrick's theorem [28] — that there were no solutions that rendered the action stationary at this order. This conclusion could be reached because the order $h^2\theta^2$ constraints on the action evaluated at the solution to the equations of motion depended only on the contributions to the field configuration that were of order $h^0\theta^0$ and $h^1\theta^1$. In the case studied here, this does not happen chiefly due to the fact that we are extremizing the Hamiltonian, which is dimensionful, rather than dimensionless as the action is, and the Higgs and gauge field have different scaling behaviours.

As suits our purposes, we shall choose the gauge $a_0 = 0$. Proceeding as in ref. [21],we shall study the behaviour of the Hamiltonian under infinitesimal dilatations of any of the (multi-)monopole solutions, $(a_i(\vec{x}), \phi(\vec{x}))$, to the noncommutative Yang-Mills-Higgs equations found in this paper, those infinitesimal dilatations preserving the boundary conditions satisfied at infinity by the (multi-)monopole solution:

$$a'_i = \lambda a_i(\lambda\vec{x}), \quad \phi' = \phi(\lambda\vec{x}), \quad \lambda = 1 + \delta\lambda. \tag{A.1}$$

The Hamiltonian for an arbitrary Seiberg-Witten map that yields an action of first order in time derivatives is given by eq. (2.4). We want to obtain the scaling properties of the different contributions to $\mathcal{H}$, taking into account eq. (A.1) and dimensional considerations. Because a_μ and ϕ scale in a different way, contributions to the Hamiltonian expanded in terms of ordinary fields at a given order in $h\theta^{\mu\nu}$ will scale differently depending on the number of ϕ fields they have. In the case of the Standard Seiberg-Witten map — for its definition, see paragraph just below eq. (2.3), it is easy to see that $A_\mu[a_\rho]$ is independent of ϕ, while Φ is linear in ϕ. This allows us to separate $\mathcal{H}$ in terms independent of ϕ and terms that are quadratic in ϕ, whose scaling behaviour is readily obtained just by using dimensional analysis. Thus, we write $\mathcal{H}^{\rm st}$ — "st" stands for standard Seiberg-Witten map — as an expansion in powers of $h\theta^{\mu\nu}$ as follows

$$\begin{aligned}
\mathcal{H}^{\rm st} &= \mathcal{H}^{\rm st}_A + \mathcal{H}^{\rm st}_\Phi,\\
\mathcal{H}^{\rm st}_A &= \mathrm{Tr}\int d^3\,\vec{x}\,B_iB_i,\\
\mathcal{H}^{\rm st}_\Phi &= \mathrm{Tr}\int d^3\,\vec{x}\,D_i\Phi D_i\Phi\\
\mathcal{H}^{\rm st}_A &= \sum_{l\geq 0} h^l\mathcal{H}^{{\rm st}(l)}_A,\\
\mathcal{H}^{\rm st}_\Phi &= \sum_{l\geq 0} h^l\,\mathcal{H}^{{\rm st}(l)}_\Phi.
\end{aligned} \tag{A.2}$$

$\mathcal{H}^{\rm st}_A$ is independent of ϕ, and $\mathcal{H}^{\rm st}_\Phi$ is quadratic in ϕ. The scaling properties of these terms are then given by

$$\delta\mathcal{H}^{{\rm st}(l)}_A = (1+2l)\delta\lambda\mathcal{H}^{{\rm st}(l)}_A, \quad \delta\mathcal{H}^{{\rm st}(l)}_\Phi = (-1+2l)\delta\lambda\mathcal{H}^{{\rm st}(l)}_\Phi. \tag{A.3}$$

When evaluating these terms in field configurations that can be written as in eq. (3.1), the following additional expansions are obtained:

$$\begin{aligned}
\mathcal{H}^{{\rm st}(l)}_{A/\Phi} &= \sum_{m\geq 0} h^m\mathcal{H}^{{\rm st}(l,m)}_{A/\Phi},\\
\mathcal{H}^{{\rm st}(l,m)}_{A/\Phi} &= \frac{1}{m!}\frac{d^m}{dh^m}\mathcal{H}^{{\rm st}(l)}_{A/\Phi}[a^{(0)}_\mu + h^k a^{(k)}_\mu, \phi^{(0)} + h^k\phi^{(k)}]|_{h=0}.
\end{aligned}$$

Therefore the invariance of $\mathcal{H}^{\rm st}$ under the infinitesimal transformations in eq. (A.1) is equivalent to:

$$\sum_n h^n[(1+2n)\mathcal{H}^{{\rm st}(n)}_A + (-1+2n)\mathcal{H}^{{\rm st}(n)}_\Phi] = 0 = \sum_{k\geq 0} h^k\sum_{l=0}^{k}[(1+2l)\mathcal{H}^{{\rm st}(l,k-l)}_A + (-1+2l)\mathcal{H}^{{\rm st}(l,k-l)}_\Phi],$$

i.e.,

$$\sum_{l=0}^{k}[(1+2l)\mathcal{H}^{{\rm st}(l,k-l)}_A + (-1+2l)\mathcal{H}^{{\rm st}(l,k-l)}_\Phi] = 0 \quad \forall k\geq 0. \tag{A.4}$$

For $k = 0$ this is equivalent to $\mathcal{H}_A^{\mathrm{st}(0,0)} - \mathcal{H}_\Phi^{\mathrm{st}(0,0)} = 0$, which is satisfied by the ordinary BPS monopoles. For $k = 1$ eq. (A.4) gives

$$\mathcal{H}_A^{\mathrm{st}(0,1)} - \mathcal{H}_\Phi^{\mathrm{st}(0,1)} + 3\mathcal{H}_A^{\mathrm{st}(1,0)} + \mathcal{H}_\Phi^{\mathrm{st}(1,0)} = 0. \tag{A.5}$$

This relation holds trivially in the SU(2) and SO(5) cases because, recalling that we are dealing with the standard Seiberg-Witten applications, the first-order-in-$h\theta^{\mu\nu}$ contributions to the field configurations are just appropriate linear combinations of the zero modes of the ordinary fields. It is also satisfied in the SU(3) case when evaluating in the field configuration $\phi = \phi_\beta^{(0)} + h\phi^s T_\beta^s$, $a_i = a_{i\beta}^{(0)} + h a_i^s T_\beta^s$, with $\phi_\beta^{(0)}$, $a_{i\beta}^{(0)}$ given by eqs. (3.14) and (3.4) and ϕ^s, a_i^s given in eqs. (4.7) and (4.9); each term in eq. (A.5) turns out to vanish, because all the traces are of the type $\mathrm{Tr}\, T_\beta^s T_\beta^a = 0$.

By substituting $k = 2$ in eq. (A.4), one obtains the following:

$$\mathcal{H}_A^{\mathrm{st}(0,2)} - \mathcal{H}_\Phi^{\mathrm{st}(0,2)} + 3\mathcal{H}_A^{\mathrm{st}(1,1)} + \mathcal{H}_\Phi^{\mathrm{st}(1,1)} + 5\mathcal{H}_A^{\mathrm{st}(2,0)} + 3\mathcal{H}_\Phi^{\mathrm{st}(2,0)} = 0.$$

In contrast with the case analysed in ref. [21], the equation involves the order $h^2\theta^2$ contributions to the field configurations, due to the fact that in eq. (A.3) the $\mathcal{H}^{(n)}$ terms scale with powers of λ that are non-zero for $n = 0$. Hence, we do not find any obstruction — implied by Derrick's theorem — to the existence, at second-order in $h\theta^{\mu\nu}$ and for $a_0 = 0$, of static solutions to the noncommutative Yang-Mills-Higgs equations.

In the case of arbitrary SW maps, the scaling behaviour of the different contributions to the Hamiltonian is more complicated, due to the fact that — see eq. (2.3) — A_i will receive contributions with arbitrary even numbers of ϕ's. Therefore, though we can always separate the terms of $\mathcal{H}$ as in eq. (A.3), with $\mathcal{H}_\Phi$ still scaling as in eq. (A.3), now $\mathcal{H}_A$ will not be independent of ϕ and the scaling behaviour will change. Nevertheless, the important issue is that there will exist terms $\mathcal{H}^{(n)}$ that will scale with powers of λ that are non-zero for $n = 0$, so that when imposing the stationarity condition at order $h^2\theta^2$, we will have again contributions of the type $\mathcal{H}^{(0,2)}$ and the same conclusion as with the standard Seiberg-Witten map will be reached.

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